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(2+1)-dimensional integrable lattice hierarchies related to discrete fourth-order nonisospectral problems

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Abstract
In this paper, we consider the (2+1)-dimensional discrete fourth-order nonisospectral problem. By using the Lax technique, three new (2+1)-dimensional nonisospectral four-field integrable lattice hierarchies are constructed. Their reductions yield three (1+1)-dimensional isospectral four-field integrable lattice hierarchies due to Mlaszak–Marciniak. We make a comparison between the (1+1)-dimensional discrete fourth-order nonisospectral problem and the third-order nonisospectral problem. We found that the integrable lattice hierarchies related to the discrete fourth-order nonisospectral problem have new characteristics.

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1. Introduction

As is well known, the investigation of multidimensional integrable systems is always an important and attractive topic. In the continuous three-dimensional case, the Kadomtsev–Petviashvili (KP) equation [1], arising in many fields of physics, such as fluid mechanics, plasma physics, etc, is an important integrable system. We also know that the self-dual Yang–Mills (SDYM) equation is of great importance in both physics and mathematics [2–4]. It has been found that three-dimensional reductions of the SDYM yield many equations including the KP, modified KP, (2+1)-dimensional N-wave, and Davey–Stewartson equations [5–7]. On the other hand, in a multidimensional discrete case, perhaps the best-known integrable systems are the (2+1)-dimensional Toda lattice and its hierarchy and the (2+1)-dimensional Volterra lattice [8, 9]. In [10], by using an r-matrix formalism, Blaszak and Marciniak (BM) constructed integrable lattice systems and their bi-Hamiltonian structure, which related to the following m-order discrete isospectral problem:

$$L_m \psi_n = \lambda \psi_n, \tag{1.1}$$

where

$$L_m = E^{\alpha+m} + u_{\alpha+m-1} E^{\alpha+m-1} + u_{\alpha+m-2} E^{\alpha+m-2} + u_\alpha E^\alpha \tag{1.2}$$

with $-m < \alpha \leqslant -1$, and the field function $u_j = u_j(n,t)$ and wave function $\psi_n = \psi(n,t)$ (here n is discrete, and t is continuous), and E is the shift operator in the variable n, defined $E^j f_n = f_{n+j}$. In [11], the m-order discrete isospectral problem is generalized to (2+1) dimensions.

We recall that most of the known integrable systems (continuous or discrete, (1+1)-dimensional or multidimensional) are related to isospectral problems. Nonisospectral scattering problems, of course since the work of Calogero [12], have continued to be the subject of much study, both in the continuous and discrete cases [13–19]. In the continuous case, the first example is due to Calogero [12], and has as a subcase the equation

$$u_{xt} = u_{xxxy} + 4u_x u_{xy} + 2u_{xx} u_y. \tag{1.3}$$

This equation arises as the compatibility condition of the linear system

$$\psi_{xx} + (u_x - \lambda)\psi = 0, \qquad \psi_t = 4\lambda\psi_y + 2u_y\psi_x - u_{xy}\psi, \tag{1.4}$$

where the spectral parameter $\lambda = \lambda(y,t)$ satisfies the constraint [13, 14]

$$\lambda_t = 4\lambda\lambda_y. \tag{1.5}$$

Since the appearance of spectral problems (1.4)–(1.5), many papers on continuous (2+1)-dimensional nonisospectral linear problems have been published (see, e.g., references in [15]). As is well known, nonisospectral linear problems are of great importance in both physics and mathematics. In physics, the variable coefficients soliton equations related to nonisospectral linear problems, such as the variable coefficient KdV, variable coefficient mKdV, variable coefficient KP, can describe nonlinear waves in non-uniformity media [20–22]. On the other hand, the Painlevé equations (continuous and discrete), arising in many field of physics including statistical mechanics, plasma physics, nonlinear waves, quantum gravity and quantum field theory, can be reviewed as stationary flows of the nonisospectral soliton equations (continuous and discrete) [23–27]. In [19], Levi and Grundland (LG) considered a nonisospectral extension of the spectral problem (1.1) with $m = 3$. The obtained lattice hierarchies related to the third-order spectral problem are the nonisospectral generalizations of the BM three-field lattice hierarchy. However, as we have known, there is little work for multidimensional discrete nonisospectral flows.

In this paper, we will concentrate on the construction of (2+1)-dimensional nonisospectral integrable discrete hierarchies. This we do by considering the following (2+1)-dimensional discrete fourth-order nonisospectral linear problem:

$$L_4 \psi_n = \lambda \psi_n, \tag{1.6}$$

where the operator L_4 takes the following form, respectively,

$$L_4 = E + u_0 + u_{-1}E^{-1} + u_{-2}E^{-2} + u_{-3}E^{-3}, \tag{1.7}$$

$$L_4 = E^2 + u_1 E + u_0 + u_{-1}E^{-1} + u_{-2}E^{-2}, \tag{1.8}$$

$$L_4 = E^3 + u_2 E^2 + u_1 E + u_0 + u_{-1}E^{-1}. \tag{1.9}$$

Here, the field function $u_j = u_j(n, t, y)$, the wave function $\psi_n = \psi(n, t, y)$, n is discrete, t and y are continuous, the spectral parameter $\lambda = \lambda(t, y)$, and $\lambda_t \neq 0$. It is obvious that the (2+1)-dimensional discrete fourth-order nonisospectral linear problem is equivalent to the following matrix form:

$$E\psi_n(\lambda) = U_n(u_n, \lambda)\psi_n(\lambda), \tag{1.10}$$

where the field function $u_n = (p_n, q_n, r_n, s_n)^T$, and the matrix $U_n(u_n, \lambda)$ is, respectively,

$$U_n(u_n, \lambda) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ p_n & q_n & r_n + \lambda & s_n \\ 1 & 0 & 0 & 0 \end{pmatrix}, \tag{1.11}$$

$$U_n(u_n, \lambda) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ p_n & q_n + \lambda & r_n & s_n \\ 1 & 0 & 0 & 0 \end{pmatrix}, \tag{1.12}$$

$$U_n(u_n, \lambda) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ p_n + \lambda & q_n & r_n & s_n \\ 1 & 0 & 0 & 0 \end{pmatrix}. \tag{1.13}$$

We assume that the time evolution of the wave function ψ_n satisfies the equation

$$\frac{\mathrm{d}\psi_n(\lambda)}{\mathrm{d}t} = \omega(\lambda)\frac{\mathrm{d}\psi_n(\lambda)}{dy} + V_n^{(m)}(u_n, \lambda)\psi_n(\lambda), \tag{1.14}$$

and the spectral parameter $\lambda = \lambda(t, y)$ satisfies a nonisospectral condition of the form

$$\lambda_t = \omega(\lambda)\lambda_y + \beta(\lambda), \tag{1.15}$$

where $\omega(\lambda)$ and $\beta(\lambda)$ are two functions to be specified. We can see that the spectral problem (1.10) and (1.14) is an extension of the discrete third-order nonisospectral problem to the (2+1)-dimensional and higher-order case and the spectral problem is also the discrete version of the (2+1)-dimensional continuous spectral problem (1.4)–(1.5). By using the compatibility condition of the system (1.10) and (1.14) with (1.15)

$$\frac{\partial U_n}{\partial t} + \beta(\lambda)\frac{\partial U_n}{\partial \lambda} - \omega(\lambda)\frac{\partial U_n}{\partial y} = V_{n+1}^{(m)}U_n - U_n V_n^{(m)}, \tag{1.16}$$

and the Lax technique, we will construct three new (2+1)-dimensional nonisospectral four-field integrable lattice hierarchies. We will show that their reductions yield (1+1)-dimensional isospectral four-field integrable lattice hierarchies due to BM. We will make a comparison between (1+1)-dimensional discrete fourth-order nonisospectral problems and third-order nonisospectral problems. We find that integrable lattice hierarchies related to the discrete fourth-order nonisospectral problems have new characteristics. We remark here that the lattice equation hierarchy derived from the linear spectral equation is integrable in the Lax sense.

2. New (2+1)-dimensional nonisospectral four-field integrable lattice hierarchies

In this section, we construct new (2+1)-dimensional nonisospectral four-field integrable lattice hierarchies by considering the (2+1)-dimensional discrete nonisospectral linear problem (1.10) and (1.14). Our aim is to seek the proper matrix $V_n^{(m)}(\lambda)=\left(v_{ij}^{(m)}(n, \lambda)\right)_{4\times 4}$ with $v_{ij} = v_{ij}(A(\lambda), B(\lambda), C(\lambda), D(\lambda))$ such that the nonisospectral discrete zero curvature equation (1.16) yields the (2+1)-dimensional integrable lattice hierarchy.

2.1. New (2+1)-dimensional nonisospectral four-field integrable lattice hierarchy related to the nonisospectral linear problem (1.10) and (1.14) with (1.11)

Let us first consider the (2+1) discrete nonisospectral linear problem (1.10) and (1.14) with (1.11). A new (2+1)-dimensional nonisospectral integrable lattice hierarchy related to the spectral problem will be constructed. A direct calculation gives

$$\begin{aligned}
&v_{11} = -(E^2+E^{-1})(pA+qB) - E^{-2}(r+\lambda)C - (E^{-2}+E^{-1}+1)sD,\\
&v_{12} = E^{-2}C - E^{-1}(c+\lambda)B, \qquad v_{13} = E^{-1}B, \qquad v_{14} = sA,\\
&v_{21} = pB + EsA, \qquad v_{22} = qB + EpA + E(v_{11} - pA),\\
&v_{23} = E^{-1}C, \qquad v_{24} = sB,\\
&v_{31} = pC + EsB, \qquad v_{32} = qC + EpB + E^2 sA,\\
&v_{33} = (r+\lambda)C + EqB + E^2 pA + E^2(v_{11} - pA), \qquad v_{34} = sC,\\
&v_{41} = E^{-3}C - E^{-2}(r+\lambda)B - E^{-1}qA,\\
&v_{42} = E^{-2}B - E^{-1}(r+\lambda)A, \qquad v_{43} = E^{-1}A,\\
&v_{44} = E^{-1}(v_{11} - pA),
\end{aligned} \tag{2.1}$$

and

$$K\bar{\wp} - \lambda J\bar{\wp} = F^{(1)}, \tag{2.2}$$

where

$$\bar{\wp} = (A(\lambda), B(\lambda), C(\lambda), D(\lambda))^T, \qquad F^{(1)} = u_t - \omega(\lambda)u_y + \beta(\lambda)\theta.$$

Here A, B, C and D are functions of the field u_n and spectral λ, and vector $\theta = (0,0,1,0)^T$, and K and J are two skew-symmetric matrix operators given by

$$K = \begin{pmatrix} k_{11} & k_{12} & k_{13} & k_{14}\\ k_{21} & k_{22} & k_{23} & k_{24}\\ k_{31} & k_{32} & k_{33} & k_{34}\\ k_{41} & k_{42} & k_{43} & k_{44} \end{pmatrix} \tag{2.3}$$

and

$$J = \begin{pmatrix} 0 & Es - sE^{-2} & p(1-E^{-2}) & 0\\ E^2 s - sE^{-1} & Ep - pE^{-1} & q(1-E^{-1}) & 0\\ (E^2-1)p & (E-1)q & 0 & (E^3-1)s\\ 0 & 0 & s(1-E^{-3}) & 0 \end{pmatrix}, \tag{2.4}$$

with

$$\begin{aligned}
&k_{11} = p(E^{-2}-E)(1+E)p - qEs + sE^{-1}q,\\
&k_{12} = p(E^{-2}-1)(E+1)q - rEs + sE^{-2}r,\\
&k_{13} = p(E^{-2}-1)r + Es - sE^{-3}, \qquad k_{14} = p(E^{-3}-1)(E+E^2+E^3)s,\\
&k_{21} = q(1-E^2)(1+E^{-1})p - rE^2 s + sE^{-1}r,\\
&k_{22} = q(E^{-1}-E)q + pE^{-1}r - rEp + E^2 s - sE^{-2},\\
&k_{23} = Ep - pE^{-2} + q(E^{-1}-1)r, \qquad k_{24} = q(1-E^3)(1+E^{-1})s,\\
&k_{31} = r(1-E^2)p + E^3 s - sE^{-1}, \qquad k_{32} = E^2 p - pE^{-1} - r(E-1)q,\\
&k_{33} = Eq - qE^{-1}, \qquad k_{34} = r(1-E^3)s, \qquad k_{41} = s(1-E^3)(E^{-1}+E^{-2}+E^{-3})s,\\
&k_{42} = s(E^{-3}-1)(1+E)q, \qquad k_{43} = s(E^{-3}-1)r,\\
&k_{44} = s(E^{-3}-E)(1+E+E^2)s.
\end{aligned} \tag{2.5}$$

By the Lax technique, we assume expansions for A, B, C and D of the form

$$A = \sum_{j=-1}^{m} a_j(n,t,y)\lambda^{m-j}, \qquad B = \sum_{j=-1}^{m} b_j(n,t,y)\lambda^{m-j},$$
$$C = \sum_{j=-1}^{m} c_j(n,t,y)\lambda^{m-j}, \qquad D = \sum_{j=-1}^{m} d_j(n,t,y)\lambda^{m-j}, \tag{2.6}$$

and for $\omega(\lambda)$ and $\beta(\lambda)$ of the form

$$\omega(\lambda) = \sum_{j=-2}^{m-1} \omega_j^{(1)}\lambda^{m-j}, \qquad \beta(\lambda) = \sum_{j=-2}^{m} \beta_j^{(1)}\lambda^{m-j}. \tag{2.7}$$

Substituting in equations (2.2), and separating different powers of λ, we obtain the equations

$$J\,(a_{-1}, b_{-1}, c_{-1}, d_{-1})^T = \omega_{-2}^{(1)}u_y - \beta_{-2}^{(1)}\theta \tag{2.8}$$

and

$$K\wp_{j-1} = J\wp_j - F_{j-1}^{(1)}, \qquad j = 0, 1, 2, \ldots, m, \tag{2.9}$$

where

$$\wp_j = (a_j, b_j, c_j, d_j)^T, \qquad F_j^{(1)} = \omega_j^{(1)}u_y - \beta_j^{(1)}\theta. \tag{2.10}$$

Then the discrete zero curvature equation (1.16) yields the following four-field lattice hierarchy:

$$(u_n)_{t_m} = X_m, \qquad m \geqslant -1, \tag{2.11}$$

where

$$X_m = K\wp_m - \beta_m^{(1)}\theta.$$

Equations (2.8)–(2.9) can be solved as

$$\begin{pmatrix} a_{-1} \\ b_{-1} \\ c_{-1} \\ d_{-1} \end{pmatrix} = \omega_{-2}^{(1)}J^{-1}u_y - \beta_{-2}^{(1)}J^{-1}\theta + \begin{pmatrix} 0 \\ 0 \\ \alpha_{-1} \\ 0 \end{pmatrix} \tag{2.12}$$

and

$$\wp_j = \omega_{j-1}^{(1)}J^{-1}u_y + J^{-1}K\wp_{j-1} - \beta_{j-1}^{(1)}J^{-1}\theta + \begin{pmatrix} 0 \\ 0 \\ \alpha_j \\ 0 \end{pmatrix}, \qquad j = 0, 1, 2, \ldots, m, \tag{2.13}$$

where J^{-1}, the inverse operator of J, is given by

$$J^{-1} = \begin{pmatrix} J_{11} & (E^2s - sE^{-1})^{-1} & 0 & J_{14} \\ (Es - sE^{-2})^{-1} & 0 & 0 & J_{24} \\ 0 & 0 & 0 & (1 - E^{-3})^{-1}s^{-1} \\ J_{41} & J_{42} & s^{-1}(E^3 - 1)^{-1} & J_{44} \end{pmatrix}. \tag{2.14}$$

Here

$$J_{11} = (E^2 s - sE^{-1})^{-1}(pE^{-1} - Ep)(Es - sE^{-2})^{-1},$$
$$J_{14} = (E^2 s - sE^{-1})^{-1}(pE^{-1} - Ep)(sE^{-2} - Es)^{-1}[p(E^2 + E)\Re s^{-1}] - (E^2 s - sE^{-1})^{-1}[q\Re E^2 s^{-1}],$$
$$J_{24} = (sE^{-2} - Es)^{-1})[p(E^2 + E)\Re s^{-1}],$$
$$J_{41} = s^{-1}\Re(E + 1)[p(sE^{-1} - E^2 s)^{-1}(pE^{-1} - Ep)(Es - sE^{-2})^{-1}] - s^{-1}\Re[q(Es - sE^{-2})^{-1}],$$
$$J_{42} = s^{-1}(E + 1)\Re[p(sE^{-1} - E^2 s)^{-1}],$$
$$J_{44} = -s^{-1}\Re[(E + 1)pJ_{14} + qJ_{24}],$$

where the operator $\Re = (E^2 + E + 1)^{-1} = \sum_{j=0}^{\infty}(E^{3j} - E^{3j+1})$. Setting $Q = KJ^{-1}$, our lattice hierarchy (2.11) can be written as

$$(u_n)_{t_m} = \sum_{j=1}^{m+2} \omega_{m-j}^{(1)} Q^j u_{n,y} + \sum_{j=0}^{m+1} \alpha_{m-j} Q^j g_n - \sum_{j=0}^{m+2} \beta_{m-j}^{(1)} Q^j \theta, \tag{2.15}$$

where

$$g_n = \begin{pmatrix} p_n(r_{n-2} - r_n) + s_{n+1} - s_n \\ p_{n+1} - p_n + q_n(r_{n-1} - r_n) \\ q_{n+1} - q_n \\ s_n(r_{n-3} - r_n) \end{pmatrix}. \tag{2.16}$$

Equation (2.15) is a new (2+1)-dimensional nonisospectral four-field integrable lattice hierarchy, which is the generalization of a (1+1)-dimensional isospectral BM four-field lattice hierarchy. The first term of the right-hand-side of the hierarchy corresponds to an extension of the BM lattice hierarchy to (2+1) dimensions; the second term consists of the standard isospectral BM four-field lattice flows. The third term consists of additional (1+1)-dimensional nonisospectral terms. It is worth remarking here that the structure of the (2+1)-dimensional nonisospectral lattice hierarchy is new and interesting. Under the reduction $\partial_y = 0$, the lattice hierarchy reduces to (1+1)-dimensional lattice hierarchy

$$(u_n)_{t_m} = \sum_{j=0}^{m+1} \alpha_{m-j} Q^j g_n - \sum_{j=0}^{m+2} \beta_{m-j}^{(1)} Q^j \theta, \tag{2.17}$$

which is a nonisospectral extension of the BM four-field lattice hierarchy. We give now the first set of equations of our lattice hierarchy (2.17). Setting $m = -1, \alpha_{-1} = 1$, and $m = 0, \alpha_{-1} = 1, \alpha_0 = 0$, and $\beta_{-2}^{(1)} = a$, we have, respectively,

$$(u_n)_t = g_n - \beta_{-1}^{(1)}\theta + ae_n^{(1)} \tag{2.18}$$

and

$$(u_n)_t = Qg_n - \left(\beta_0^{(1)} + aQ^2\right)\theta + \beta_{-1}^{(1)} e_n^{(1)}, \tag{2.19}$$

where

$$e_n^{(1)} = \begin{pmatrix} 3p_n \\ 2q_n \\ r_n \\ 4s_n \end{pmatrix}, \qquad Qg_n = \begin{pmatrix} p_n(q_{n-2} + q_{n-1} - q_n - q_{n+1} - r_{n-2}^2 + r_n^2) \\ +s_n(r_{n-3} + r_{n-2}) - s_{n+1}(r_n + r_{n+1}) \\ q_n(q_{n-1} - q_{n+1} - r_{n-1}^2 + r_n^2) + p_n(r_{n-2} + r_{n-1}) \\ -p_{n+1}(r_n + r_{n+1}) + s_{n+2} - s_n \\ p_{n+2} - p_n - q_{n+1}(r_n + r_{n+1}) + q_n(r_{n-1} + r_n) \\ s_n(q_{n-3} + q_{n-2} - q_n - q_{n+1} - r_{n-3}^2 + r_n^2) \end{pmatrix}.$$

Under the reductions $\partial_y = 0$ and $\lambda_t = 0$, the last two equations lead to the first equations of the (1+1)-dimensional isospectral B-M four-field lattice hierarchy.

2.2. New (2+1)-dimensional nonisospectral four-field integrable lattice hierarchy related to the nonisospectral linear problem (1.10) and (1.14) with (1.12)

In this subsection, we will discuss the (2+1) discrete nonisospectral linear problem (1.10) and (1.14) with (1.12). Another new (2+1)-dimensional nonisospectral four-field integrable lattice hierarchy related to the spectral problem will be obtained. First, we have

$$\begin{aligned}
&v_{11} = -E^{-1}[(q+\lambda)B + (E+1)^{-1}(pA + rC) + sD],\\
&v_{12} = E^{-2}C - E^{-1}rB, \qquad v_{13} = E^{-1}B, \qquad v_{14} = sA,\\
&v_{21} = pB + EsA, \qquad v_{22} = -E^{-1}[(E+1)^{-1}(pA + rC) + sD], \qquad v_{23} = E^{-1}C,\\
&v_{24} = sB, \qquad v_{31} = pC + EsB, \qquad v_{32} = (q+\lambda)C + EpB + E^2sA,\\
&v_{33} = (E+1)^{-1}(rC - EpA) - E^{-1}sD, \qquad v_{34} = sC,\\
&v_{41} = E^{-3}C - E^{-2}rB - E^{-1}(q+\lambda)A, \qquad v_{42} = E^{-2}B - E^{-1}rA,\\
&v_{43} = E^{-1}A, \qquad v_{44} = -E^{-2}[(E + (E+1)^{-1})pA + (q+\lambda)B + sD + (E+1)^{-1}rC]
\end{aligned} \tag{2.20}$$

and

$$H\wp - \lambda P\wp = F^{(2)}, \tag{2.21}$$

where

$$\wp = (D(\lambda), A(\lambda), B(\lambda), C(\lambda))^T, \qquad F^{(2)} = v_t - \omega(\lambda)v_y + \beta(\lambda)\theta.$$

Here $v = (s_n, p_n, q_n, r_n)^T$, and H and P are two skew-symmetric matrix operators described as

$$H = \begin{pmatrix} s(E^{-2} - E^2)s & s(E^{-2} - E)p & s(E^{-2} - 1)q & s(E^{-2} - E^{-1})r \\ p(E^{-1} - E^2)s & h_{22} & p(E^{-1} - 1)q + sE^{-2}r - rEs & h_{24} \\ q(1 - E^2)s & q(1-E)p - rE^2s + sE^{-1}r & E^2s - sE^{-2} + pE^{-1}r - rEp & Ep - pE^{-2} \\ r(E - E^2)s & h_{42} & E^2p - pE^{-1} & h_{44} \end{pmatrix} \tag{2.22}$$

with

$$\begin{aligned}
h_{22} &= p(E+1)^{-1}(E^{-1} - E^2)p - qEs + sE^{-1}q,\\
h_{24} &= Es - sE^{-3} + p(E^{-1} - 1)(E+1)^{-1}r,\\
h_{42} &= E^3s - sE^{-1} + r(E - E^2)(E+1)^{-1}p,\\
h_{44} &= Eq - qE^{-1} + r(E-1)(E+1)^{-1}r
\end{aligned}$$

and

$$P = \begin{pmatrix} 0 & 0 & s(1 - E^{-2}) & 0 \\ 0 & Es - sE^{-1} & p(1 - E^{-1}) & 0 \\ (E^2 - 1)s & (E-1)p & 0 & 0 \\ 0 & 0 & 0 & E^{-1} - E \end{pmatrix}. \tag{2.23}$$

Assuming the same expansions for A, B, C, D, described by equation (2.6), and ω and β given by equation (2.7) with the replace $\omega_j^{(1)} \to \omega_j^{(2)}$ and $\beta_j^{(1)} \to \beta_j^{(2)}$, and then separating different powers of λ, we obtain the equations

$$P\,(d_{-1}, a_{-1}, b_{-1}, c_{-1})^T = \omega_{-2}^{(2)}v_y - \beta_{-2}^{(2)}\theta \tag{2.24}$$

and

$$H\wp_{j-1} = P\wp_j - F_{j-1}^{(2)}, \qquad j = 0, 1, 2, \ldots, m, \tag{2.25}$$

where

$$\wp_j = \left(d_j, a_j, b_j, c_j\right)^T, \qquad F_j^{(2)} = \omega_j^{(2)} v_y - \beta_j^{(2)}\theta. \tag{2.26}$$

Then the discrete zero curvature equation (1.16) yields the following four-field lattice hierarchy:

$$(u_n)_{t_m} = Y_m, \qquad m \geqslant -1, \tag{2.27}$$

where

$$Y_m = H\wp_m - \beta_m^{(2)}\theta.$$

We solve equations (2.24)–(2.25) as

$$\begin{pmatrix} d_{-1} \\ a_{-1} \\ b_{-1} \\ c_{-1} \end{pmatrix} = \omega_{-2}^{(2)} P^{-1} v_y - \frac{1}{2} s^{-1}\beta_{-2}^{(2)} n \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \alpha_{-1} \\ \gamma_{-1} \end{pmatrix} \tag{2.28}$$

and

$$\wp_j = \omega_{j-1}^{(2)} P^{-1} v_y + P^{-1} H\wp_{j-1} - \frac{1}{2} s^{-1}\beta_{j-1}^{(2)} n \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \alpha_j \\ \gamma_j \end{pmatrix}, \qquad j = 0, 1, 2, \ldots, m, \tag{2.29}$$

where

$$P^{-1} = \begin{pmatrix} p_{11} & s^{-1}(E+1)^{-1}[p(sE^{-1} - Es)^{-1}] & s^{-1}(E^2 - 1)^{-1} & 0 \\ (sE^{-1} - Es)^{-1}[p(1 + E^{-1})^{-1}s^{-1}] & (Es - sE^{-1})^{-1} & 0 & 0 \\ (1 - E^{-2})^{-1}s^{-1} & 0 & 0 & 0 \\ 0 & 0 & 0 & (E^{-1} - E)^{-1} \end{pmatrix} \tag{2.30}$$

with

$$p_{11} = s^{-1}(E+1)^{-1}[p(Es - sE^{-1})^{-1}](p(E^{-1} + 1)^{-1}s^{-1}).$$

Setting $W = HP^{-1}$, our lattice hierarchy (2.27) can be written as

$$(v_n)_{t_m} = \sum_{j=1}^{m+2} \omega_{m-j}^{(2)} W^j v_{n,y} + \sum_{j=0}^{m+1} (\alpha_{m-j} W^j \phi_n + \gamma_{m-j} W^j \chi_n) - \sum_{j=0}^{m+2} \beta_{m-j}^{(2)} W^j \theta, \tag{2.31}$$

where

$$\phi_n = \begin{pmatrix} s_n(q_{n-2} - q_n) \\ p_n(q_{n-1} - q_n) + s_n r_{n-2} - r_n s_{n+1} \\ s_{n+2} - s_n + p_n r_{n-1} - r_n p_{n+1} \\ p_{n+2} - p_n \end{pmatrix}, \tag{2.32}$$

$$\chi_n = \begin{pmatrix} s_n(r_{n-2} - r_{n-1}) \\ s_{n+1} - s_n + p_n(E+1)^{-1}(r_{n-1} - r_n) \\ p_{n+1} - p_n \\ q_{n+1} - q_n + r_n(E+1)^{-1}(r_{n+1} - r_n) \end{pmatrix}. \tag{2.33}$$

The structure of the (2+1) nonisospectral lattice hierarchy is the same as that of the lattice hierarchy (2.15). Under the reduction $\partial_y = 0$, the lattice hierarchy leads to another (1+1)-dimensional nonisospectral BM four-field lattice hierarchy:

$$(v_n)_{t_m} = \sum_{j=0}^{m+1}(\alpha_{m-j} W^j \phi_n + \gamma_{m-j} W^j \chi_n) - \sum_{j=0}^{m+2} \beta_{m-j}^{(2)} W^j \theta. \tag{2.34}$$

Let us write down the first members of our lattice hierarchy (2.34). Setting $m = -1, \alpha_{-1} = 1, \gamma_{-1} = 0$ or $m = -1, \alpha_{-1} = 0, \gamma_{-1} = 1$, and $\beta_{-2}^{(2)} = 2b$, we have, respectively,

$$(v_n)_t = \phi_n - \beta_{-1}^{(2)}\theta + be_n^{(2)} \tag{2.35}$$

and

$$(v_n)_t = \chi_n - \beta_{-1}^{(2)}\theta + be_n^{(2)}, \tag{2.36}$$

where $e_n^{(2)} = (4s_n, 3p_n, 2q_n, r_n)^T$. Under the reductions $\lambda_t = 0$, the last two equations lead to the first equations of another (1+1)-dimensional isospectral BM four-field lattice hierarchy.

2.3. New (2+1)-dimensional nonisospectral four-field integrable lattice hierarchy related to the nonisospectral linear problem (1.10) and (1.14) with (1.13)

In this subsection, the third new (2+1)-dimensional nonisospectral four-field integrable lattice hierarchy related to nonisospectral linear problem (1.10) and (1.14) with (1.13) will be derived. From the discrete zero curve equation, we have

$$\begin{aligned}
&v_{11} = -\Re[(E+1)qB + rC + sD], \qquad v_{12} = E^{-2}C - E^{-1}rB,\\
&v_{13} = E^{-1}B, \qquad v_{14} = sA, \qquad v_{21} = (p+\lambda)B + EsA, \qquad v_{22} = qB + EV_{11},\\
&v_{23} = E^{-1}C, \qquad v_{24} = sB, \qquad v_{31} = (p+\lambda)C + EsB,\\
&v_{32} = E^2 sA + E(p+\lambda)B + qC, \qquad v_{33} = EqB + rC + E^2 V_{11}, \qquad v_{34} = sC,\\
&v_{41} = E^{-3}C - E^{-2}rB - E^{-1}qA, \qquad v_{42} = E^{-2}B - E^{-1}rA,\\
&v_{43} = E^{-1}A, \qquad v_{44} = E^{-1}V_{11} - E^{-1}(p+\lambda)A
\end{aligned} \tag{2.37}$$

and

$$L\wp - \lambda M\wp = F^{(3)}, \tag{2.38}$$

where

$$L = \begin{pmatrix} s(E^{-1}-E^3)\Re s & s(E^{-1}-1)p & s(E^{-1}-E)\Re q & s(E^{-1}-1)\Re r \\ p(1-E)s & sE^{-1}q - qEs & sE^{-2}r - rEs & Es - sE^{-3} \\ q(E-E^3)\Re s & sE^{-1}r - rE^2 s & E^2 s - sE^{-2} + pE^{-1}r - rEp + q(E^2-1)\Re q & Ep - pE^{-2} + q(E-1)\Re r \\ r(E^2-E^3)\Re s & E^3 s - sE^{-1} & E^2 p - pE^{-1} + r(E^2-E)\Re q & Eq - qE^{-1} + r(E^2-1)\Re r \end{pmatrix}, \tag{2.39}$$

$$M = \begin{pmatrix} 0 & s(1-E^{-1}) & 0 & 0 \\ (E-1)s & 0 & 0 & 0 \\ 0 & 0 & rE - E^{-1}r & E^{-2} - E \\ 0 & 0 & E^{-1} - E^2 & 0 \end{pmatrix}, \tag{2.40}$$

and

$$F^{(3)} = v_t - \omega(\lambda)v_y + \beta(\lambda)\zeta,$$

with $\zeta = (0, 1, 0, 0)^T$. Assuming expansions for A, B, C, and D of the form given by (2.6) and $\omega(\lambda)$ and $\beta(\lambda)$ of the form given by (2.7) with the change $\omega_j^{(1)} \to \omega_j^{(3)}$ and $\beta_j^{(1)} \to \beta_j^{(3)}$, and separating different powers of λ, we obtain the following equations:

$$M\,(d_{-1}, a_{-1}, b_{-1}, c_{-1})^T = \omega_{-2}^{(3)} u_y - \beta_{-2}^{(3)} \zeta \tag{2.41}$$

and

$$L\wp_{j-1} = M\wp_j - F_{j-1}^{(3)}, \qquad j = 0, 1, 2, \ldots, m. \tag{2.42}$$

We can see that the discrete zero curvature equation (1.16) yields the following four-field lattice hierarchy:

$$(v_n)_{t_m} = Z_m, \qquad m \geqslant -1, \tag{2.43}$$

where

$$Z_m = L\wp_m - \beta_m^{(3)} \zeta.$$

Equations (2.41)–(2.42) can be solved as

$$(d_{-1}, a_{-1}, b_{-1}, c_{-1})^T = e_{-1} + \omega_{-2}^{(3)} M^{-1} v_y - s^{-1} \beta_{-2}^{(3)} n\,(1, 0, 0, 0)^T, \tag{2.44}$$

where $e_{-1} = (0, \mu_{-1}, \gamma_{-1}, \eta_{-1} + \gamma_{-1}\Re r_{n+1})^T$, and

$$\wp_j = M^{-1} L\wp_{j-1} + e_j + \omega_{j-1}^{(3)} M^{-1} v_y - s^{-1} \beta_{j-1}^{(3)} n\,(1, 0, 0, 0)^T, \qquad j = 0, 1, 2, \ldots, m, \tag{2.45}$$

with $e_j = (0, \mu_j, \gamma_j, \eta_j + \gamma_j \Re r_{n+1})^T$. Here the matrix operator M^{-1} has the form

$$M^{-1} = \begin{pmatrix} 0 & s^{-1}(E-1)^{-1} & 0 & 0 \\ E(E-1)^{-1}s^{-1} & 0 & 0 & 0 \\ 0 & 0 & 0 & (1-E^3)^{-1}E \\ 0 & 0 & (1-E^3)^{-1}E^2 & (1-E^3)^{-1}E^2(rE - E^{-1}r)(E^3-1)^{-1}E \end{pmatrix}, \tag{2.46}$$

Setting $R = LM^{-1}$, we can write the lattice hierarchy (2.43) as

$$(v_n)_{t_m} = \sum_{j=1}^{m+2} \omega_{m-j}^{(3)} R^j v_{n,y} + \sum_{j=0}^{m+1} R^j L e_{m-j} - \sum_{j=0}^{m+2} \beta_{m-j}^{(3)} R^j \zeta. \tag{2.47}$$

Under the reduction $\partial_y = 0$, it yields the third new (1+1)-dimensional nonisospectral B-M four-field lattice hierarchy:

$$(v_n)_{t_m} = \sum_{j=0}^{m+1} R^j L e_{m-j} - \sum_{j=0}^{m+2} \beta_{m-j}^{(3)} R^j \zeta. \tag{2.48}$$

Let us write down the first members of our lattice hierarchies (2.47) and (2.48). Setting $m = -1$, and $\beta_{-2}^{(3)} = 3c$, we have, respectively,

$$(v_n)_t = L e_{-1} + \omega_{-2}^{(3)} R v_{n,y} + f_n \tag{2.49}$$

and

$$(v_n)_t = L e_{-1} + f_n, \tag{2.50}$$

where

$$f_n = c e_n^{(2)} - \beta_{-1}^{(3)} \zeta.$$

Furthermore, choosing $\mu_{-1} = 1, \gamma_{-1} = \eta_{-1} = 0$, or $\mu_{-1} = \gamma_{-1} = 0, \eta_{-1} = 1$, or $\mu_{-1} = \eta_{-1} = 0, \gamma_{-1} = 1$, and noting $(E^2 + E + 1)^{-1}n = (n-1)/3$, we obtain the following three lattice equation, respectively,

$$(v_n)_t = \begin{pmatrix} s_n(p_{n-1} - p_n) \\ q_{n-1}s_n - q_n s_{n+1} \\ r_{n-1}s_n - r_n s_{n+2} \\ s_{n+3} - s_n \end{pmatrix} + f_n, \tag{2.51}$$

$$(v_n)_t = \begin{pmatrix} s_n\Re(r_{n-1} - r_n) \\ s_{n+1} - s_n \\ p_{n+1} - p_n + q_n\Re(r_{n+1} - r_n) \\ q_{n+1} - q_n + r_n\Re(r_{n+2} - r_n) \end{pmatrix} + f_n \tag{2.52}$$

and

$$(v_n)_t = \begin{pmatrix} s_n\Re(q_{n-1} - q_{n+1}) + s_n(E-1)\Re(r_{n-1}\Re r_n) \\ r_{n-2}s_n - r_n s_{n+1} + s_{n+1}\Re r_{n+2} - s_n\Re r_{n-2} \\ p_n r_{n-1} - r_n p_{n+1} - s_n + s_{n+2} + q_n(E-1)\Re[(E+1)q_n + r_n\Re r_{n+1}] \\ -p_n\Re r_{n-1} + p_{n+1}\Re r_{n+2} \\ p_{n+2} - p_n + r_n\Re(E^2-1)(q_n + r_n\Re r_{n+1}) \\ +q_{n+1}\Re r_{n+2} - q_n\Re r_n \end{pmatrix} + f_n. \tag{2.53}$$

Under the isospectral case, the last three equations become the first seed equations of the third (1+1)- isospectral BM four-field lattice hierarchy.

3. A comparison between the discrete fourth-order spectral problem and the discrete third-order spectral problem

In this section, we make a comparison between the discrete fourth-order spectral problem and the discrete third-order spectral problem. First we note that in the special case $s_n = 0$, the discrete fourth-order spectral problem (1.10) with (1.11) and (1.12) reduces to the corresponding discrete third-order spectral problem, and thus the obtained (2+1) four-field nonisospectral lattice hierarchies (2.15) and (2.31) reduces to (2+1) three-field nonisospectral lattice hierarchies, respectively. However, the discrete fourth-order spectral problem (1.10) with (1.13) cannot reduce to a proper discrete third-order spectral problem. Let us consider the first members (2.18) and (2.35) of (1+1)-dimensional nonisospectral lattice hierarchies (2.17) and (2.34). In the special $s_n = 0$, the two four-field lattice equations reduce to the following three-field lattice equations, respectively,

$$\begin{aligned} \dot{p}_n &= p_n(r_{n-2} - r_n + 3a) \\ \dot{q}_n &= p_{n+1} - p_n + q_n(r_{n-1} - r_n + 2a) \\ \dot{r}_n &= q_{n+1} - q_n + ar_n - \beta_{-1}^{(1)} \end{aligned} \tag{3.1}$$

and

$$\begin{aligned} \dot{p}_n &= p_n(q_{n-1} - q_n + 3b) \\ \dot{q}_n &= p_n r_{n-1} - p_{n+1}r_n + 2bq_n - \beta_{-1}^{(2)} \\ \dot{r}_n &= p_{n+2} - p_n + br_n. \end{aligned} \tag{3.2}$$

Applying the transformation

$$
\begin{aligned}
p_n &= -\mathrm{e}^{\alpha_{n+1}-\alpha_n-\alpha_{n-1}+\alpha_{n-2}} \\
q_n &= -\frac{\mathrm{d}^2\alpha_n}{\mathrm{d}t^2} + a\frac{\mathrm{d}\alpha_n}{\mathrm{d}t} - \frac{3}{4}a^2 n(n-1) + \beta_{-1}^{(1)} n \\
r_n &= \frac{\mathrm{d}\alpha_n}{\mathrm{d}t} - \frac{\mathrm{d}\alpha_{n+1}}{\mathrm{d}t} + \frac{3}{2}an
\end{aligned}
\tag{3.3}
$$

to equation (3.1) and the transformation

$$
\begin{aligned}
p_n &= \mathrm{e}^{\alpha_{n-1}-2\alpha_n+\alpha_{n+1}} \\
q_n &= \frac{\mathrm{d}\alpha_n}{\mathrm{d}t} - \frac{\mathrm{d}\alpha_{n+1}}{\mathrm{d}t} + 3bn \\
r_n &= \left(\frac{\mathrm{d}^2\alpha_{n+1}}{\mathrm{d}t^2} - 2b\frac{\mathrm{d}\alpha_{n+1}}{\mathrm{d}t} + 3b^2 n(n+1) - \beta_{-1}^{(2)}(n+1)\right)\mathrm{e}^{2\alpha_{n+1}-\alpha_n-\alpha_{n+2}}
\end{aligned}
\tag{3.4}
$$

to equation (3.2), we obtain, respectively,

$$
\begin{aligned}
\frac{\mathrm{d}^3\alpha_n}{\mathrm{d}t^3} - \frac{3}{2}a\frac{\mathrm{d}^2\alpha_n}{\mathrm{d}t^2} + \frac{1}{2}a^2\frac{\mathrm{d}\alpha_n}{\mathrm{d}t} &= \frac{3}{8}a^3 n(n-1) - \frac{1}{2}a\beta_{-1}^{(1)} n \\
&\quad + \left(\frac{\mathrm{d}^2\alpha_n}{\mathrm{d}t^2} - a\frac{\mathrm{d}\alpha_n}{\mathrm{d}t} + \frac{3}{4}a^2 n(n-1) - \beta_{-1}^{(1)} n\right)\left(\frac{\mathrm{d}\alpha_{n+1}}{\mathrm{d}t} - 2\frac{\mathrm{d}\alpha_n}{\mathrm{d}t} + \frac{\mathrm{d}\alpha_{n-1}}{\mathrm{d}t}\right) \\
&\quad + \mathrm{e}^{\alpha_{n+2}-\alpha_{n+1}-\alpha_n+\alpha_{n-1}} - \mathrm{e}^{\alpha_{n+1}-\alpha_n-\alpha_{n-1}+\alpha_{n-2}}
\end{aligned}
\tag{3.5}
$$

and

$$
\begin{aligned}
\frac{\mathrm{d}^3\alpha_n}{\mathrm{d}t^3} - 3b\frac{\mathrm{d}^2\alpha_n}{\mathrm{d}t^2} + 2b^2\frac{\mathrm{d}\alpha_n}{\mathrm{d}t} &= 3b^3 n(n-1) - \beta_{-1}^{(2)} bn \\
&\quad + \left(\frac{\mathrm{d}^2\alpha_n}{\mathrm{d}t^2} - 2b\frac{\mathrm{d}\alpha_n}{\mathrm{d}t} + 3b^2 n(n-1) - \beta_{-1}^{(2)} n\right)\left(\frac{\mathrm{d}\alpha_{n+1}}{\mathrm{d}t} - 2\frac{\mathrm{d}\alpha_n}{\mathrm{d}t} + \frac{\mathrm{d}\alpha_{n-1}}{\mathrm{d}t}\right) \\
&\quad + \mathrm{e}^{\alpha_{n+2}-\alpha_{n+1}-\alpha_n+\alpha_{n-1}} - \mathrm{e}^{\alpha_{n+1}-\alpha_n-\alpha_{n-1}+\alpha_{n-2}}.
\end{aligned}
\tag{3.6}
$$

The two equations are higher-order differential-difference equations of the Toda type with an n-dependent coefficient. Under the isospectral case, the two equations yield a equation of the Toda type:

$$
\frac{\mathrm{d}^3\alpha_n}{\mathrm{d}t^3} = \frac{\mathrm{d}^2\alpha_n}{\mathrm{d}t^2}\left(\frac{\mathrm{d}\alpha_{n+1}}{\mathrm{d}t} - 2\frac{\mathrm{d}\alpha_n}{\mathrm{d}t} + \frac{\mathrm{d}\alpha_{n-1}}{\mathrm{d}t}\right) + \mathrm{e}^{\alpha_{n+2}-\alpha_{n+1}-\alpha_n+\alpha_{n-1}} - \mathrm{e}^{\alpha_{n+1}-\alpha_n-\alpha_{n-1}+\alpha_{n-2}}
\tag{3.7}
$$

This conclusion is the same as that observed by LG in [19]. However, for the nonisospectral case, equation (3.5) is different from equation (3.6), though in the special case $b = a/2, \beta_{-1}^{(1)} = \beta_{-1}^{(2)}$, the two equations are same. Let us turn to four-field nonisospectral lattice equations (2.18), (2.35), and (2.51). We find that the structures of the four-field nonisospectral lattice equations are very complicated. In fact, it is very difficult to write the four-field nonisospectral lattice equations as evolution equations for the one-field $\alpha_n(t)$. Applying the transformation

$$
\begin{aligned}
s_n &= \mathrm{e}^{\alpha_{n-2}-\alpha_{n-3}+\alpha_n-\alpha_{n+1}} \\
r_n &= \frac{\mathrm{d}\alpha_{n+1}}{\mathrm{d}t} - \frac{\mathrm{d}\alpha_n}{\mathrm{d}t} + \frac{4}{3}an \\
q_n &= \frac{\mathrm{d}^2\alpha_n}{\mathrm{d}t^2} - a\frac{\mathrm{d}\alpha_n}{\mathrm{d}t} - \frac{2}{3}a^2 n(n-1) + \beta_{-1}^{(1)} n \\
p_n &= (E-1)^{-1}\left[\frac{\mathrm{d}^3\alpha_n}{\mathrm{d}t^3} - \frac{5}{3}a\frac{\mathrm{d}^2\alpha_n}{\mathrm{d}t^2} + \frac{2}{3}a^2\frac{\mathrm{d}\alpha_n}{\mathrm{d}t} + \frac{4}{9}a^3 n(n-1) - \frac{2}{3}a\beta_{-1}^{(1)} n\right. \\
&\quad \left. + \left(\frac{\mathrm{d}^2\alpha_n}{\mathrm{d}t^2} - a\frac{\mathrm{d}\alpha_n}{\mathrm{d}t} - \frac{2}{3}a^2 n(n-1) + \beta_{-1}^{(1)} n\right)\left(\frac{\mathrm{d}\alpha_{n+1}}{\mathrm{d}t} - 2\frac{\mathrm{d}\alpha_n}{\mathrm{d}t} + \frac{\mathrm{d}\alpha_{n-1}}{\mathrm{d}t}\right)\right]
\end{aligned}
\tag{3.8}
$$

to equation (2.18) and the transformation

$$
\begin{aligned}
s_n &= \mathrm{e}^{\alpha_{n-1}-2\alpha_n+\alpha_{n+1}}\\
p_n &= \frac{\mathrm{d}\alpha_n}{\mathrm{d}t}-\frac{\mathrm{d}\alpha_{n+1}}{\mathrm{d}t}+4cn\\
q_n &= \left(\frac{\mathrm{d}^2\alpha_{n+1}}{\mathrm{d}t^2}-3c\frac{\mathrm{d}\alpha_{n+1}}{\mathrm{d}t}+6c^2n(n+1)-\beta_{-1}^{(3)}n\right)\mathrm{e}^{2\alpha_{n+1}-\alpha_n-\alpha_{n+2}}\\
r_n &= \mathrm{e}^{\alpha_{n+1}+\alpha_{n+2}-\alpha_n-\alpha_{n+3}}(E-1)^{-1}\left[\left(2c+\frac{\mathrm{d}\alpha_{n+1}}{\mathrm{d}t}-2\frac{\mathrm{d}\alpha_{n+2}}{\mathrm{d}t}+\frac{\mathrm{d}\alpha_{n+3}}{\mathrm{d}t}\right)\left(\frac{\mathrm{d}^2\alpha_{n+2}}{\mathrm{d}t^2}\right.\right.\\
&\quad\left.\left.-3c\frac{\mathrm{d}\alpha_{n+2}}{\mathrm{d}t}+6c^2(n+2)(n+1)-\beta_{-1}^{(3)}(n+1)\right)-\frac{\mathrm{d}^3\alpha_{n+2}}{\mathrm{d}t^3}+3c\frac{\mathrm{d}^2\alpha_{n+2}}{\mathrm{d}t^2}\right]
\end{aligned}
\tag{3.9}
$$

to equation (2.51), we have, respectively,

$$
\begin{aligned}
&(E-1)^{-1}\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\mathrm{d}^3\alpha_n}{\mathrm{d}t^3}-\frac{5}{3}a\frac{\mathrm{d}^2\alpha_n}{\mathrm{d}t^2}+\frac{2}{3}a^2\frac{\mathrm{d}\alpha_n}{\mathrm{d}t}+\frac{4}{9}a^3n(n-1)-\frac{2}{3}a\beta_{-1}^{(1)}n\right.\\
&\qquad\left.+\left(\frac{\mathrm{d}^2\alpha_n}{\mathrm{d}t^2}-a\frac{\mathrm{d}\alpha_n}{\mathrm{d}t}-\frac{2}{3}a^2n(n-1)+\beta_{-1}^{(1)}n\right)\left(\frac{\mathrm{d}\alpha_{n+1}}{\mathrm{d}t}-2\frac{\mathrm{d}\alpha_n}{\mathrm{d}t}+\frac{\mathrm{d}\alpha_{n-1}}{\mathrm{d}t}\right)\right]\\
&\quad=\left(\frac{\mathrm{d}\alpha_{n-1}}{\mathrm{d}t}-\frac{\mathrm{d}\alpha_{n-2}}{\mathrm{d}t}+\frac{\mathrm{d}\alpha_n}{\mathrm{d}t}-\frac{\mathrm{d}\alpha_{n+1}}{\mathrm{d}t}+\frac{1}{3}a\right)\\
&\qquad\times(E-1)^{-1}\left[\frac{\mathrm{d}^3\alpha_n}{\mathrm{d}t^3}-\frac{5}{3}a\frac{\mathrm{d}^2\alpha_n}{\mathrm{d}t^2}+\frac{2}{3}a^2\frac{\mathrm{d}\alpha_n}{\mathrm{d}t}+\frac{4}{9}a^3n(n-1)-\frac{2}{3}a\beta_{-1}^{(1)}n\right.\\
&\qquad\left.+\left(\frac{\mathrm{d}^2\alpha_n}{\mathrm{d}t^2}-a\frac{\mathrm{d}\alpha_n}{\mathrm{d}t}-\frac{2}{3}a^2n(n-1)+\beta_{-1}^{(1)}n\right)\left(\frac{\mathrm{d}\alpha_{n+1}}{\mathrm{d}t}-2\frac{\mathrm{d}\alpha_n}{\mathrm{d}t}+\frac{\mathrm{d}\alpha_{n-1}}{\mathrm{d}t}\right)\right]\\
&\qquad+\mathrm{e}^{\alpha_{n-1}-\alpha_{n-2}+\alpha_{n+1}-\alpha_{n+2}}-\mathrm{e}^{\alpha_{n-2}-\alpha_{n-3}+\alpha_n-\alpha_{n+1}}
\end{aligned}
\tag{3.10}
$$

and

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left[\mathrm{e}^{\alpha_{n+1}+\alpha_{n+2}-\alpha_n-\alpha_{n+3}}(E-1)^{-1}\left[\left(2c+\frac{\mathrm{d}\alpha_{n+1}}{\mathrm{d}t}-2\frac{\mathrm{d}\alpha_{n+2}}{\mathrm{d}t}+\frac{\mathrm{d}\alpha_{n+3}}{\mathrm{d}t}\right)\left(\frac{\mathrm{d}^2\alpha_{n+2}}{\mathrm{d}t^2}\right.\right.\right.\\
&\qquad\left.\left.\left.-3c\frac{\mathrm{d}\alpha_{n+2}}{\mathrm{d}t}+6c^2(n+2)(n+1)-\beta_{-1}^{(3)}(n+1)\right)-\frac{\mathrm{d}^3\alpha_{n+2}}{\mathrm{d}t^3}+3c\frac{\mathrm{d}^2\alpha_{n+2}}{\mathrm{d}t^2}\right]\right]\\
&\quad=c\,\mathrm{e}^{\alpha_{n+1}+\alpha_{n+2}-\alpha_n-\alpha_{n+3}}(E-1)^{-1}\left[\left(2c+\frac{\mathrm{d}\alpha_{n+1}}{\mathrm{d}t}-2\frac{\mathrm{d}\alpha_{n+2}}{\mathrm{d}t}+\frac{\mathrm{d}\alpha_{n+3}}{\mathrm{d}t}\right)\left(\frac{\mathrm{d}^2\alpha_{n+2}}{\mathrm{d}t^2}\right.\right.\\
&\qquad\left.\left.-3c\frac{\mathrm{d}\alpha_{n+2}}{\mathrm{d}t}+6c^2(n+2)(n+1)-\beta_{-1}^{(3)}(n+1)\right)-\frac{\mathrm{d}^3\alpha_{n+2}}{\mathrm{d}t^3}+3c\frac{\mathrm{d}^2\alpha_{n+2}}{\mathrm{d}t^2}\right]\\
&\qquad+\mathrm{e}^{\alpha_{n+2}-2\alpha_{n+3}+\alpha_{n+4}}-\mathrm{e}^{\alpha_{n-1}-2\alpha_n+\alpha_{n+1}}.
\end{aligned}
\tag{3.11}
$$

In the isospectral case, the last two equations reduce to, respectively,

$$
\begin{aligned}
&(E-1)^{-1}\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\mathrm{d}^3\alpha_n}{\mathrm{d}t^3}+\frac{\mathrm{d}^2\alpha_n}{\mathrm{d}t^2}\left(\frac{\mathrm{d}\alpha_{n+1}}{\mathrm{d}t}-2\frac{\mathrm{d}\alpha_n}{\mathrm{d}t}+\frac{\mathrm{d}\alpha_{n-1}}{\mathrm{d}t}\right)\right]\\
&\quad=\left(\frac{\mathrm{d}\alpha_{n-1}}{\mathrm{d}t}-\frac{\mathrm{d}\alpha_{n-2}}{\mathrm{d}t}+\frac{\mathrm{d}\alpha_n}{\mathrm{d}t}-\frac{\mathrm{d}\alpha_{n+1}}{\mathrm{d}t}\right)\\
&\qquad\times(E-1)^{-1}\left[\frac{\mathrm{d}^3\alpha_n}{\mathrm{d}t^3}+\frac{\mathrm{d}^2\alpha_n}{\mathrm{d}t^2}\left(\frac{\mathrm{d}\alpha_{n+1}}{\mathrm{d}t}-2\frac{\mathrm{d}\alpha_n}{\mathrm{d}t}+\frac{\mathrm{d}\alpha_{n-1}}{\mathrm{d}t}\right)\right]\\
&\qquad+\mathrm{e}^{\alpha_{n-1}-\alpha_{n-2}+\alpha_{n+1}-\alpha_{n+2}}-\mathrm{e}^{\alpha_{n-2}-\alpha_{n-3}+\alpha_n-\alpha_{n+1}}
\end{aligned}
\tag{3.12}
$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\{\mathrm{e}^{\alpha_{n+1}+\alpha_{n+2}-\alpha_n-\alpha_{n+3}}(E-1)^{-1}\left[\frac{\mathrm{d}^2\alpha_{n+2}}{\mathrm{d}t^2}\left(\frac{\mathrm{d}\alpha_{n+1}}{\mathrm{d}t}-2\frac{\mathrm{d}\alpha_{n+2}}{\mathrm{d}t}+\frac{\mathrm{d}\alpha_{n+3}}{\mathrm{d}t}\right)-\frac{\mathrm{d}^3\alpha_{n+2}}{\mathrm{d}t^3}\right]\right\}$$
$$= \mathrm{e}^{\alpha_{n+2}-2\alpha_{n+3}+\alpha_{n+4}} - \mathrm{e}^{\alpha_{n-1}-2\alpha_n+\alpha_{n+1}} \qquad (3.13)$$

We can see that the nonisospectral lattice equations (3.10) and (3.11) and isospectral lattice equations (3.12) and (3.13) are nonlocal. As for the four-field nonisospectral lattice equation (2.35), we have not found its form of one-field $\alpha_n(t)$, though its reduction is a variable coefficient Toda-type equation. We thus conclude that the discrete four-order spectral problem is more complicated than the discrete three-order spectral problem.

4. Conclusions

In this paper, we have discussed completely (2+1)-dimensional discrete fourth-order nonisospectral problems and constructed three new (2+1)-dimensional nonisospectral four-field integrable lattice hierarchies, which generalize (1+1)-dimensional isospectral BM four-field integrable lattice hierarchies to (2+1) dimensions and nonisospectral case. By making a comparison between the (1+1)-dimensional discrete fourth-order nonisospectral problem and discrete the third-order nonisospectral problem, we found that lattice hierarchies related to the fourth-order nonisospectral problem have new characteristics. We will further investigate the obtained (2+1)-dimensional nonisospectral lattice hierarchies including their physical applications, and the other integrability, such as the Hamiltonian structures, infinitely many conservation laws and soliton solutions.

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